Analytical investigation of the open boundary conditions in the Nagel-Schreckenberg model

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We theoretically investigate the mechanism of the open boundary conditions in the deterministic Nagel-Schreckenberg model, which was studied mainly by numerical simulations before. Our studies concentrate on the open boundaries and have found an effective approach for deducing the analytical expression of inflow. We also raise a removal rule which is analyzable. These findings provide a theoretical explanation of the behaviors of the open boundaries and allow the exact prediction of the traffic state by using the injection rate and the removal rate.

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I. INTRODUCTION

Cellular automata (CA) are a powerful tool for researching traffic flow [1–7]. In comparison with the traditional approaches, CA can be used very efficiently for computer simulations.

The Nagel-Schreckenberg (NS) model [6] is a typical one-dimensional CA model that has been researched widely for years. In this model, space and time are discrete. The road is modeled as a one-dimensional lattice with L sites that are labeled in sequence by $1-L$ from the left. Each site can be either empty or occupied by a car with velocity $0-v_{\max}$ ($v_{\max}$ is assumed to be the maximum velocity that one car can reach). All sites are updated according to the following four rules:

(1) acceleration—$v:=\min(v+1,v_{\max})$

(2) slowdown—$v:=\min(v,g)$ (where g is the number of empty sites in front of the car)

(3) randomization—$v:=\max(v-1,0)$ with probability p

(4) movement—move v sites forward, $x:=x+v$

The NS model provides a realistic description of traffic flow, and it can reproduce the basic features of real traffic flow, e.g., the phase transition between free flow and jamming [1,7]. There also exists an improved version of the NS model, called velocity-dependent randomization (VDR) model [8]. In this model, the randomization parameter p is a function of velocity v, which could be described as $p=p(v(t))$. This model can reproduce more complex phenomena, such as metastable states, which have been observed in real traffic flow [9] but do not exist in the NS model.

Behaviors of the NS (VDR) model also depend on the boundary conditions. There are two kinds of boundary conditions: periodic boundary conditions and open boundary conditions [10–13]. In a periodic system, cars move on a ring and the car density of the system remains constant. Open boundary conditions, on the other hand, are much more complex. In an open system, cars enter the road from the left boundary, and leave via the right boundary. Consequently, some characteristics that have gained much attention in open boundary systems do not exist in periodic systems, such as inflow and outflow. Moreover other characteristics of open boundary systems are quite different from those of periodic systems [12,13], including global density and density profiles.

Open boundary conditions consist of the injection rule (left boundary) and the removal rule (right boundary), which are characterized by the injection rate α and the removal rate β, respectively. As for the NS model, there are mainly two types of open boundaries: the "standard" open boundaries [12] and the "expanded" open boundaries [14]. The difference between these two types lies in the injection rule, but their removal rule is the same:

Removal rule. If a car's velocity is large enough to move out of the road from right boundary, then it is removed from the road with probability β. With probability $1-\beta$, the car stops at the last site.

This rule can be easily realized with the use of an additional site next to the right boundary. When updated, this site is cleared first and then occupied with probability $1-\beta$.

The standard injection rule directly derives from the asymmetric simple exclusion process (ASEP) model. Behaviors of the NS model with the standard injection rule have been investigated vividly in Refs. [12,13] by numerical simulations, including inflow, density profiles, and phase diagram. The standard injection rule is defined in the following way:

Standard injection rule. With probability α a car with velocity $v=v_{\max}$ is created at site 0; this car immediately moves according to the NS rules. If site 1 is occupied by another car, the injected car is deleted.

The standard injection rule is also easy to implement. However, its behaviors are very astonishing—the state of maximum current is totally absent, which has been clearly demonstrated in Refs. [12–14]. In other words, the inflow is not a monotone increasing function of the injection rate α in the system with $v_{\max}\geq 3$, which is contrary to intuition. Numerical simulations indicated that the inflow reaches its maximum ($q_{\text{in}}\approx 0.683$) at $\alpha\approx 0.83$ for $v_{\max}=5$. This characteristic causes part of possible system states to be inaccessible, and the phase diagram of the NS model is quite different from that of the ASEP model [12–14].

To overcome this disadvantage, a different injection rule is proposed [15]. It is defined as follows:

Expanded injection rule. The left boundary is expanded from a single site to a minisystem of width $v_{\max}+1$ as shown in Fig. 1. When updated, if there is a car in the minisystem, it has to be emptied first. Then a vehicle with a initial velocity of $v_{\max}$ is inserted with probability p_{in}. Its initial position is the site at the right end point of the boundary if no car is

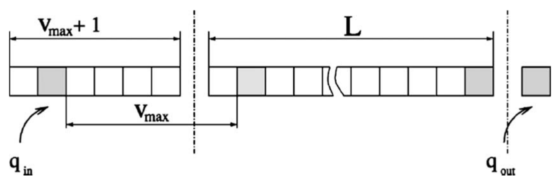

FIG. 1. The expanded injection rule proposed in Ref. [14].

present in the main system within the first $v_{\max}$ sites. Otherwise its initial position is the site with $v_{\max}$ distance from the first car in the main system (Fig. 1).

With the help of such a rule, high inflow can be achieved, so the whole spectrum of possible system states is accessible. Moreover, it is analyzable; that is, one can calculate the inflow using an expression. As a result of those advantages, the expanded injection rule has become more popular than the standard one [16–18].

However, the behaviors of the standard injection rule are much more interesting. It is significant to give an explanation of its surprising characteristics, and some conclusions have already been drawn. In Ref. [12], a phenomenon named "buffer effect" is noticed, which is developed due to the hindrance an injected car feels from the front car at the beginning of the road. It is regarded as the main cause of the strange behaviors of the standard injection rule in the system with $v_{\max} \geq 3$. Moreover, buffer effect is also used to predict the phase diagram of the system as a qualitative approach.

The buffer effect is a good qualitative explanation for the mechanism of the standard open boundary conditions. But it is better to find some analytical results for the influence of the open boundaries. Our work can be summarized into two parts. First, a method for deriving the analytical expressions of inflow is found by modeling the evolution of the system as a Markov chain, and this method is also adaptable for the expanded injection rule. Second, we design an analyzable removal rule. All these findings help to construct a system whose phase (free flow or jamming) is exactly predictable simply with α and β.

This paper is organized as follows. In Sec. II, we model the car-injection procedure, and propose a method which can derive the analytical expression of inflow. In Sec. III, the method is verified by examples. In Sec. IV, we put out an analyzable removal rule. Section V is the conclusion and discussion.

II. MODELING THE CAR-INJECTION PROCEDURE

The relationship between α and inflow is the most important characteristic of the injection rule. The α-inflow curve of the standard injection rule was previously found out by using numerical method in Ref. [12]. This section describes a method by which the analytical expression of inflow can be obtained.

Suppose that c is a car on the road; let $v_t(c)$ be the velocity of c at time t. According to the NS rules, if $\min[v_t(c)+1, v_{\max}] > g_t^+(c)$, where $g_t^+(c)$ is the number of empty sites in front of c at time t, then c can only move $g_t^+(c)$ sites ahead during the update at time t. Then c has to slow down because it cannot accelerate or keep the maximum velocity, and this procedure is called a *slowdown.* We call $g_t^+(c)$ the degree of this slowdown. The degree of a slowdown tells the car's velocity after it has slowed down.

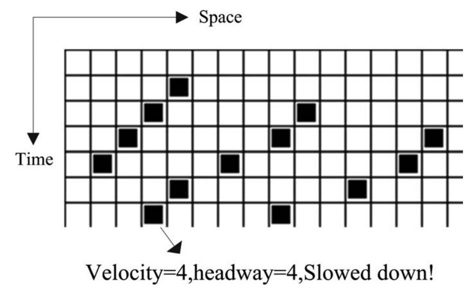

FIG. 2. Illustration of the slowdown caused by injection procedure. The vertical direction is time and the horizontal direction is space; cars move from left to right.

The influence of the removal rule is the primary cause of slowdowns [12]. But it is possible for slowdowns to happen due to the only influence of the standard injection rule, which is shown in Fig. 2 (in the deterministic NS model with $v_{\max}=5$). This kind of slowdown is named as "injection-produced slowdown" (IPSD) to distinguish it from the slowdowns caused by the removal rule. It is the topic on which we concentrate in this section.

Theorem 1. In deterministic Nagel-Schreckenberg model (DNS), the minimum degree of IPSD is 4.

Proof. See Appendix A.

Theorem 1 implies several things as follows:

(1) Suppose that c is a car on the road. If c had IPSD at time t, then $v_{t+1}(c) \geq 4$.

(2) In DNS($v_{\max} \leq 4$) (DNS with $v_{\max} \leq 4$), IPSDs never happen.

(3) Because the velocity of a car at site d has to be lower than d, a car at site 1–3 will not have IPSD.

Let c_t^0 be the car that is nearest to the left boundary at time t and $l_t(c)$ be the location of car c at time t. Then the *state of system* at time t is defined as $\min(l_t(c_t^0), v_{\max}+1)$, and denoted by S_t. Since cars on road will move and a new car may be injected at each step, the system will jump to a new state at time $t+1$, which is called "state transition." The state of the system is updated each time step. Thus the time evolution of the system can be described by a sequence of states.

Using Theorem 1, the following theorem was proved:

Theorem 2. In DNS, suppose that the state of the system at time t is S_t and no car is injected, then there is only one possible value for S_{t+1} if $v_{\max} \leq 5$.

Proof. See Appendix B.

This theorem implies that in DNS($v_{\max} \leq 5$), for two different time steps t_1 and t_2, if $S_{t_1}=S_{t_2}$ and no car is injected at times t_1 and t_2, then $S_{t_1+1} \equiv S_{t_2+1}$. Otherwise, if $v_{\max} \geq 6$, S_{t_1+1} may not be equal to S_{t_2+1}.

Moreover, if the state at time t is S_t and a car is injected, then S_{t+1} can be calculated exactly and easily according to the NS rules and the standard injection rule. We can find that there is also only one possible value for S_{t+1}:

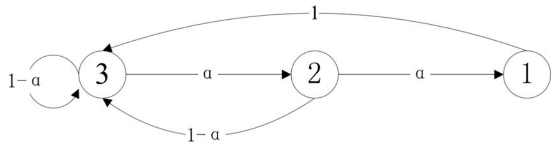

FIG. 3. The state transition diagram of DNS($v_{\max}$=2).

$$S_{t+1}=\begin{cases}3 & \text{if } S_t=1\\ S_t-1 & \text{if } S_t>1\end{cases}\quad \text{if a car is injected at time } t.$$

According to Theorem 2, in DNS with $v_{\max}\leq 5$, suppose that $S_t=i_t$ $(i_t\neq 1)$ at an arbitrary time t, if no car is injected at time t, then S_{t+1} has only one possible value i_{t+1}. Recall that the injection rate is α; the probability that no car is injected at time t is $1-\alpha$. Then

$$P(S_{t+1}=i_{t+1}|S_1=i_1,\ldots,S_t=i_t)=P(S_{t+1}=i_{t+1}|S_t=i_t)=1-\alpha.$$

If a car is injected at time t, then the following is true:

$$P(S_{t+1}=i_t-1|S_1=i_1,\ldots,S_t=i_t)=P(S_{t+1}=i_t-1|S_t=i_t)$$
$$=\alpha,\ \text{where } S_t\neq 1.$$

Especially, if $S_t=1$, whether cars are injected or not, $S_{t+1}=3$,

$$P(S_{t+1}=3|S_t=1)=1.$$

Therefore, we can assert that the discrete time evolution of the system is a homogeneous Markov chain. Moreover, if $\alpha\neq 0,1$, then this Markov chain has the following characteristics:

(1) Its state space is irreducible, because its state space is closed and every state is accessible by others.

(2) It is aperiodic, because the probability of the state transition $v_{\max}+1\rightarrow v_{\max}+1$ is $1-\alpha$.

(3) Its state space is finite, because $v_{\max}$ is a finite number.

In accordance with the theory of stochastic processes, a homogeneous Markov chain satisfying three conditions above is ergodic, whose limiting distribution exists. Let p_i denote the limiting probability of state i, and $\vec{P}=[p_1\cdots p_n]$, then p_i can be obtained by solving the equation

$$\begin{cases}\vec{P}=\vec{P}T,\\ \sum_{j=1}^{n}p_j=1,\end{cases}$$

where T is the one-step transition probability matrix, which indicates the probabilities of all possible one-step state transitions.

Actually, the inflow Q_{in} is the expectation of the number of injected cars per step. With the consideration that if the state of the system is 1, then no car could be injected. Thus the inflow with injection rate α can be calculated by using the expression $Q_{\text{in}}(\alpha)=\alpha\sum_{i=2}^{v_{\max}+1}p_i$. Noting that $\sum_{i=1}^{v_{\max}+1}p_i=1$, we have $Q_{\text{in}}(\alpha)=\alpha(1-p_1)$.

From the above analysis, the procedure for deriving the analytical expression of inflow in DNS($v_{\max}\leq 5$) can be described as follows:

Step 1. For each state, find which state it jumps to under conditions "car in" and "no car in."

Step 2. Construct the one-step transition probability matrix.

Step 3. Calculate limiting probabilities of state p_1.

Step 4. Calculate the inflow using $Q_{\text{in}}(\alpha)=\alpha(1-p_1)$, with $\alpha\neq 0,1$.

Step 5. Deal with the special case $\alpha=0,1$ (Sec. III will indicate how to do this).

It should be noticed that for the standard injection rule, expressions of inflow are different for different values of $v_{\max}$.

III. EXAMPLES AND ANALYSIS

This section demonstrates the applications of the method proposed in Sec. II. It is composed of three parts: the applications of the method to DNS($v_{\max}\leq 5$) for the exact solution, the applications of the method to DNS($v_{\max}\geq 6$) for the approximate solution, and the use of this method to solve the expanded injection rule.

A. Exact solution of inflow in DNS($v_{\max}\leq 5$)

In this subsection, the method will be applied to two cases: DNS($v_{\max}$=2) and DNS($v_{\max}$=5). The exact solutions of their inflows are achieved.

The first example is DNS($v_{\max}$=2). Its state transition diagram is given in Fig. 3.

Here is the one-step transition probability matrix:

$$T=\begin{bmatrix}0 & 0 & 1\\ \alpha & 0 & 1-\alpha\\ 0 & \alpha & 1-\alpha\end{bmatrix}.$$

If $\alpha!=0,1$ and letting $\vec{P}=[p_1,p_2,p_3]$, $\vec{P}$ could be obtained by solving

$$\begin{cases}\vec{P}=\vec{P}T,\\ p_1+p_2+p_3=1.\end{cases}$$

Then

$$\vec{P}=\left[\frac{\alpha^2}{1+\alpha+\alpha^2},\frac{\alpha}{1+\alpha+\alpha^2},\frac{1}{1+\alpha+\alpha^2}\right],$$

TABLE I. Data gained by the analytical and numerical approaches in DNS($v_{\max}$=2).

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	1
Numerical	0.099	0.194	0.281	0.359	0.429	0.490	0.543	0.590	0.631	0.663	0.667
Analytical	0.101	0.194	0.281	0.355	0.431	0.490	0.544	0.589	0.631	0.663	0.667

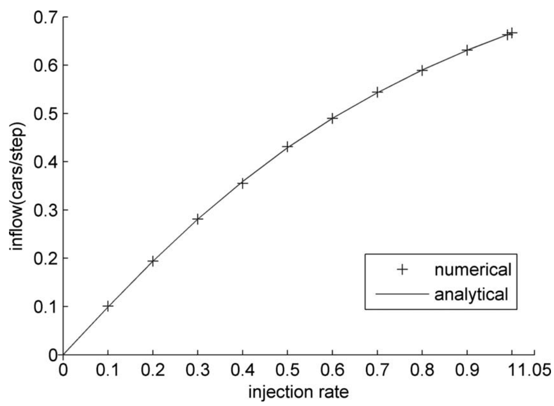

FIG. 4. Comparison between analytical and numerical results of inflow in DNS($v_{\max}$=2).

$$Q_{\text{in}}(\alpha) = \alpha(1-p_1) = \frac{\alpha(1+\alpha)}{1+\alpha+\alpha^2}, \quad \text{where} \quad \alpha \in (0,1).$$

There is a special case when $\alpha=1$. Under this condition, state transition is a cycle: $2\rightarrow 1\rightarrow 0\rightarrow 2\rightarrow\cdots$; evidently, $p_1=1/3 \Rightarrow Q_{\text{in}}=2/3$. Moreover if $\alpha=0$, $Q_{\text{in}}=0$. The same values are obtained if 0 and 1 are substituted into $Q_{\text{in}}(\alpha)$, so that the expression $Q_{\text{in}}(\alpha)=\alpha(1+\alpha)/(1+\alpha+\alpha^2)$ holds for the whole domain of α.

Hence, we know that in DNS($v_{\max}$=2),

$$Q_{\text{in}}(\alpha) = \frac{\alpha(1+\alpha)}{1+\alpha+\alpha^2}, \quad \alpha \in [0,1].$$

Table I and Fig. 4 show that data gained by numerical simulations are highly consistent with the predictions of analytical expression.

The second example is DNS($v_{\max}$=5). Its state transition diagram is given in Fig. 5.

The result is shown as follows (details are omitted):

$$Q_{\text{in}}(\alpha) = \frac{\alpha(1+\alpha-\alpha^3+\alpha^4)}{1+\alpha-\alpha^3+\alpha^4+\alpha^5}.$$

Table II and Fig. 6 show that the analytical results exactly agree well with those by numerical simulations. From the examples above, we can see that for DNS with $v_{\max}\leq 5$, the exact solution of inflow is found.

B. Approximate solution of inflow in DNS($v_{\max}\geq 6$)

In most of the researches and applications, $v_{\max}$ is limited below 10, because a greater number has no practical significance. This method can also find out the approximation to the inflow if $v_{\max}$ is between 6 and 10. This subsection will give examples of cases DNS($v_{\max}$=7), DNS($v_{\max}$=8), and DNS($v_{\max}$=10).

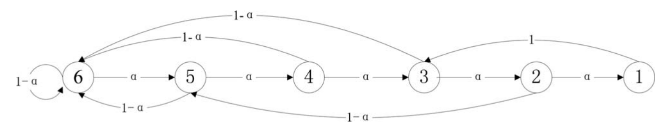

FIG. 5. The state transition diagram of DNS($v_{\max}$=5).

Theorem 2 tells us that in DNS($v_{\max}$=7), for two different steps t_1 and t_2 that satisfy $S_{t_1}=S_{t_2}$ and if no car is injected at times t_1 and t_2, then S_{t_1+1} may differ from S_{t_2+1}. That is, the time evolution of DNS($v_{\max}$=7) is not Markovian. For example, we can give two state sequences:

State sequence 1. $7\overset{1}{\rightarrow}6\overset{1}{\rightarrow}5\overset{1}{\rightarrow}4\overset{1}{\rightarrow}3\overset{1}{\rightarrow}2\overset{0}{\rightarrow}6$.

State sequence 2. $7\overset{1}{\rightarrow}6\overset{1}{\rightarrow}5\overset{1}{\rightarrow}4\overset{1}{\rightarrow}3\overset{1}{\rightarrow}2\overset{1}{\rightarrow}1\overset{0}{\rightarrow}2\overset{0}{\rightarrow}5$.

In the sequences above, condition "1" stands for car in, whereas condition "0" stands for no car in. It is clear that if $S_t=2$ and no car is injected, then S_{t+1} may be 5 or 6.

However, if we randomly choose a value from all the possible values of S_{t+1} and assume it to be the only possible value for S_{t+1}, then we can still apply the same method. In this case, if we assume that the only possible value for S_{t+1} is 6, then the one-step transition probability matrix is obtained:

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 1-\alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1-\alpha \end{bmatrix}.$$

Comparison of results by two approaches is shown in Table III and Fig. 7.

Analyses for DNS($v_{\max}$=8) are more complex. In addition to the exceptions $2\overset{0}{\rightarrow}6$ and $2\overset{0}{\rightarrow}5$, there are also $3\overset{0}{\rightarrow}7$ and $3\overset{0}{\rightarrow}8$. Letting the assumption be that $S_t=2\overset{0}{\rightarrow}S_{t+1}=6$ and $S_t=3\overset{0}{\rightarrow}S_{t+1}=8$, then the analytical expression could be deduced. Comparison of results by two approaches is shown in Table IV and Fig. 8.

As for DNS($v_{\max}$=10), the results are shown in Table V and Fig. 9. We can find that the results gained by analytical method are not as accurate as those in the previous cases.

For larger value of $v_{\max}$, the evolution of the system is much more complex; thus more assumptions have to be

TABLE II. Data gained by the analytical and numerical approaches in DNS($v_{\max}$=5).

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	1
Numerical	0.098	0.201	0.301	0.402	0.490	0.570	0.633	0.670	0.681	0.669	0.667
Analytical	0.099	0.200	0.299	0.397	0.489	0.571	0.633	0.670	0.680	0.669	0.667

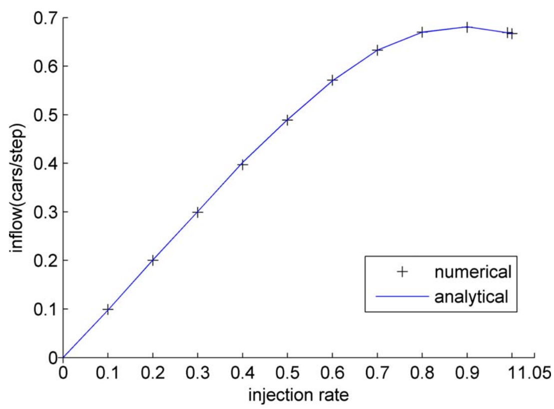

FIG. 6. (Color online) Comparison between analytical and numerical results of inflow in DNS($v_{\max}=5$).

made to apply this method. That will evidently make the deviation between the numerical and analytical methods higher. But, as for DNS with $v_{\max}\leq 10$, the comparison shows that a high degree of accuracy is achieved.

C. Solving the expanded injection rule

In fact, the analytical expression of inflow under the expanded injection rule was already derived in Ref. [15]:

$$Q_{\text{in}}(\alpha)=\frac{\alpha-\alpha^{v_{\max}+1}}{1-\alpha^{v_{\max}+1}}.$$

The same expression can also be obtained by the presently described method. Here the state of the system is also defined as $S_t=\min(l_t(c_t^0),v_{\max}+1)$. In accordance with the expanded injection rule (already defined in Sec. I), we can find that all cars on the road move with velocities $v_{\max}$ without the influence of removal rule. Then the following can be easily proved to be true:

$$S_{t+1}=\begin{cases}S_t-1 & \text{if a car is injected at time } t\\ v_{\max}+1 & \text{if no car is injected at time } t\end{cases}$$
$$\text{if } S_t\neq 1,$$

$$S_{t+1}=v_{\max}+1 \quad \text{if } S_t=1.$$

It is easy to know that its time evaluation is a Markov chain, and this Markov chain is also ergodic because it satisfies the three conditions mentioned in Sec II. The state transition diagram of the system with expanded injection rule is shown in Fig. 10.

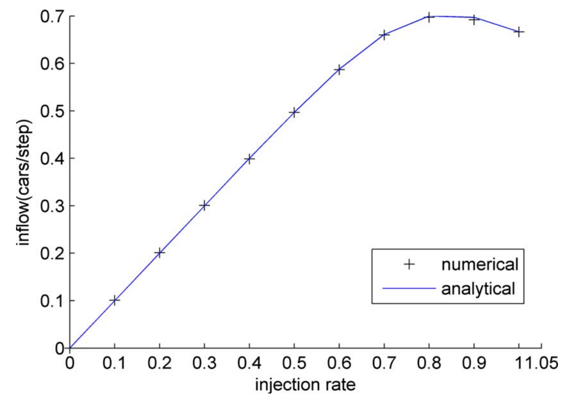

FIG. 7. (Color online) Comparison between analytical and numerical results of inflow in DNS($v_{\max}=7$).

The one-step transition probability matrix is

$$T=\underbrace{\begin{bmatrix} & & & 1\\ \alpha & & & 1-\alpha\\ & \ddots & & \vdots\\ & & \alpha & 1-\alpha\\ & & & \alpha \;\; 1-\alpha\end{bmatrix}}_{v_{\max}+1}.$$

Letting $\vec{P}=[p_1\cdots p_{v_{\max}+1}]$, we get $p_1=(\alpha^{v_{\max}}-\alpha^{v_{\max}+1})/(1-\alpha^{v_{\max}+1})$ by solving the equation

$$\begin{cases}\vec{P}=\vec{P}T,\\ \displaystyle\sum_{j=1}^{v_{\max}+1}p_j=1.\end{cases}$$

Because if the state of the system is 1, then no car can enter the road. Thus the inflow can also be calculated by $Q_{\text{in}}(\alpha)=\alpha\sum_{i=2}^{v_{\max}+1}p_i=\alpha(1-p_1)$. Then $Q_{\text{in}}(\alpha)=(\alpha-\alpha^{v_{\max}+1})/(1-\alpha^{v_{\max}+1})$. This result is the same as that in Ref. [15], but the deduction is simpler.

IV. ANALYZABLE REMOVAL RULE

The standard removal rule is quite simple and demonstrates the reality. However, the outflow under the standard rule is also unpredictable by analytical means.

Suppose that the flow in the road is q, and the removal rate is β. The procedure of a car's departure is composed of two steps. The first step is to enter the boundary site (the last site) with probability q, and the second is to leave the boundary site with probability β. Thus, the outflow could be di-

TABLE III. Data gained by the analytical and numerical approaches in DNS($v_{\max}=7$).

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Numerical	0.100	0.200	0.300	0.400	0.497	0.588	0.661	0.700	0.697	0.6667
Analytical	0.101	0.201	0.301	0.399	0.497	0.587	0.660	0.697	0.692	0.6667

TABLE IV. Data gained by the analytical and numerical approaches in DNS(v_{max}=8).

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Numerical	0.100	0.200	0.300	0.400	0.499	0.593	0.671	0.712	0.702	0.6667
Analytical	0.099	0.201	0.300	0.395	0.498	0.589	0.668	0.707	0.696	0.6667

vided into two parts: the inflow into the boundary site and the outflow from the boundary site.

In a statistical view, if $q=\beta$, then the long-time average inflow of the boundary site is supposed to be equal to the outflow. However, as a result of the existence of stochastic fluctuations, the inflow is always different from the outflow in a short period of time. When the inflow gets larger than the outflow, cars will gather at the right boundary, which causes traffic jams. But as has already been proved [16], the outflow of jams is the maximum flow that can be reached in the NS model. This means it is very probable that new jams will be generated by the already formed jams, which will push the system into the jamming phase. Thus, in the phase diagram, the point (α,β) satisfying $Q_{in}(\alpha)=\beta$ is in the region of the jamming phase [12,14,15].

In order to construct a predictable system, we can use a simple but effective method. The right boundary shown in Fig. 11 is the expansion of the standard right boundary. The single boundary site is enlarged into a minisystem. We can consider this minisystem as a car parking with limited parking space. The removal rule is composed of two steps:

(1) If a car is to move out of the main road, it will then be inserted into the parking if the parking is not full. Otherwise, this car stops at the last site of the main road.

(2) With probability β, one car (this car could be selected by the "first in first out" rule) in the parking moves out of the system immediately.

Obviously, if the capacity of the parking is set to 1, the original right boundary is recovered. However, with more parking space, cars that cannot leave in time could be stored in the parking temporarily. In this way the stochastic fluctuations can be "neutralized," which means that a short-time shock wave of car flow will not push the system into the jamming phase, as shown previously. Generally speaking, a larger capacity results in better performance but greater loss of reality. Numerical simulations show that the capacity $C=\max(3,v_{max})$ is a good choice.

Next, an analysis of the expanded right boundary is given. If the inflow $Q_{in}<\beta$, then the parking will hardly be full, and cars can enter and leave it freely. Thus, traffic flow will be in the free flow phase, and the outflow that from the parking is equal to Q_{in}. If $Q_{in}\geq\beta$, then the parking will always be full of cars; thus the newly arrived cars will have to stop at the end of the main road, which results in jams, and the outflow from the car parking is limited to β.

Therefore, the analytical expression of the outflow under the expanded right boundary condition is given as follows:

$$Q_{out}=\begin{cases} Q_{in} & \text{if } Q_{in}<\beta \\ \beta & \text{if } Q_{in}\geq\beta. \end{cases}$$

This result is confirmed by numerical simulations (Fig. 12).

Hence, the state of traffic flow is predictable by comparing α and β. If $Q_{in}(\alpha)>\beta$, then it will be jamming phase. Otherwise, traffic will be in the free flow phase. This implies that the jamming phase and the free phase are separated by the α-Q_{in} line, just as shown in Fig. 13.

V. CONCLUSION AND DISCUSSION

This study has investigated the mechanism of the standard open boundaries in the deterministic Nagel-Schreckenberg model in an analytical way. Our work mainly focused on two things. First, we modeled the car-injection procedure under standard injection rule by Markov chain, which made the deduction of analytical expression of inflow possible. Furthermore the applications of our method were also demonstrated. Second, we put out an analyzable removal rule.

All of these works make the phase of traffic flow predictable simply by knowing the injection rate α and the removal

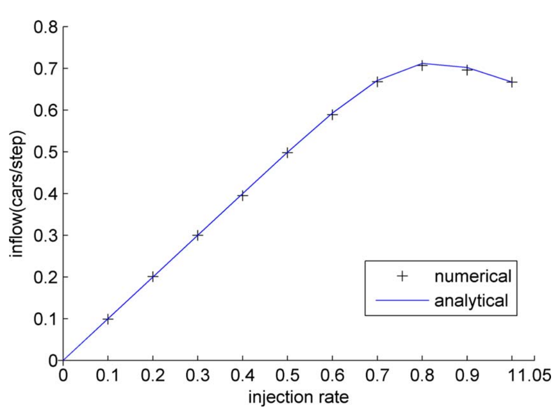

FIG. 8. (Color online) Comparison between analytical and numerical results of inflow in DNS(v_{max}=8).

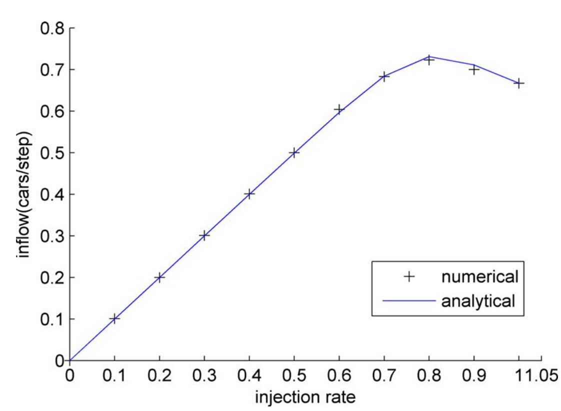

FIG. 9. (Color online) Comparison between analytical and numerical results of inflow in DNS(v_{max}=10).

TABLE V. Data gained by the analytical and numerical approaches in DNS($v_{\max}$=10).

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Analytical	0.100	0.200	0.300	0.400	0.499	0.597	0.684	0.731	0.711	0.6667
Numerical	0.101	0.200	0.301	0.401	0.500	0.604	0.683	0.723	0.700	0.6667

rate β. They also allow deeper insight into the open boundary conditions.

However, the investigation reveals only a very small part of NS model. After all, the randomized parameter p is without consideration, which is an important parameter that determines the bulk dynamics. In the stochastic NS model, characteristics of IPSD will become much more complex and thus may not be described by a few theorems. However, for the stochastic NS model, if $v_{\max}$, α, and p are very small, the method can still work. This is illustrated by the case in which $v_{\max}=2$.

As a result of the randomization, the time evolution of the stochastic NS model with $v_{\max}=2$ is evidently not Markovian. To apply the method proposed in Sec. II, we have to make some assumptions; the state transition diagram is shown in Fig. 14 (details are omitted). The solution is

$$Q_{\text{in}}(\alpha)=\alpha\left(1-\frac{\alpha(\alpha p-\alpha-p)}{\alpha^2 p+\alpha p-\alpha^2-\alpha-\alpha p^2-1}\right).$$

Note that if $p=0$, the expression above is the one for deterministic NS model with $v_{\max}=2$ (see Sec. III).

Figure 15 shows the deviation between analytical results and simulation results for different combinations of p and α. It can be clearly seen that increase in either p or α makes the deviation greater.

From this simple illustration, it can be deduced that for even the simplest case of the stochastic model, the application of the method presented in this paper is fussy and the results have low precision. Hence, with regard to the stochastic model, statistical approaches may be more practicable and effective.

ACKNOWLEDGMENTS

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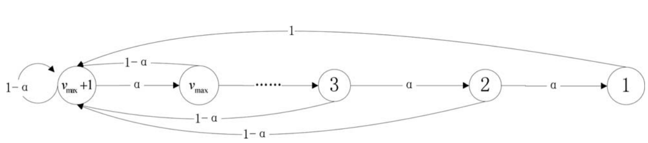

FIG. 10. The state transition diagram of DNS with expanded left boundary.

APPENDIX A: PROOF OF THEOREM 1

Lemma A1. If $v_t(c)<v_{\max}$ and $g_t^+(c)\le v_t(c)$, then $g_{t+1}^+[n^-(c)]\ge g_t^+(c)$, where $n^-(c)$ is the first car on the left side of c.

Proof. Since $g_t^+(c)\le v_t(c)$, then $l_{t+1}(c)=l_t(c)+g_t^+(c)$. Letting $c'=n^-(c)$, $l_{t+1}(c')\le l_t(c)-1$ implies that $g_{t+1}^+(c')=l_{t+1}(c)-l_{t+1}(c')-1\ge[l_t(c)+g_t^+(c)]-[l_t(c)-1]-1\ge g_t^+(c)$.

Lemma A2. If $v_t(c)<v_{\max}$ and $g_t^+(c)>v_t(c)$, then $g_{t+1}^+[n^-(c)]>v_t(c)$.

Proof. Under the given condition, car c is able to increase its velocity; thus $l_{t+1}(c)=l_t(c)+v_t(c)+1$. Letting $c'=n^-(c)$, $l_{t+1}(c')\le l_t(c)-1$ implies that $g_{t+1}^+(c')=l_{t+1}(c)-l_{t+1}(c')-1>v_t(c)$.

Theorem 1. In DNS, the minimum degree of IPSD is 4.

Proof. This theorem can be proved by two steps.

There exists 4-IPSD in DNS if $v_{\max}>4$, which is already illustrated in Fig. 2. Thus, we have to prove that s-IPSD($s<4$) cannot be produced by injection procedure.

Supposing that c' experiences s-IPSD at time t' and letting $n^+(c)$ be the first car on the right side of c, then we can find another car c after applying the following:

Procedure
Let $c=n^+(c')$ $t=t'-1$
while (c is slowed down at time t)
{
$c=n^+(c)$ $t=t-1$
}
return c

We can definitely find a car c that satisfies the condition, because the first injected car always remains free driving. Therefore, Lemmas A1 and A2 imply that $v_t(c)<s$. Obviously we know that 0-IPSD is impossible in DNS because it causes $v_t(c)<0$.

Suppose that c is injected into the system at time t_i ($t_i<t$), then there are only three possible ways for c to arrive at $l_t(c)$ at time t:

(1) "Direct hit"—$t-t_i=1$ and $v_t(c)=l_t(c)$.

(2) "Acceleration"—$t-t_i>1$ and $\forall\,\bar{t}$ $t_i\le\bar{t}-1<\bar{t}\le t$; $v_{\bar{t}}=\min(v_{\bar{t}-1}+1,v_{\max})$. That means c remains free driving from time t.

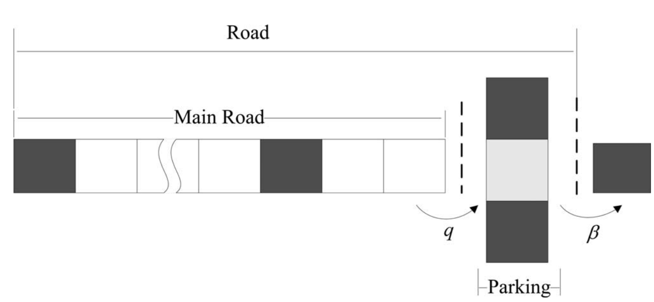

FIG. 11. Schematic representation of the enlarged right boundary; the single boundary site is expanded.

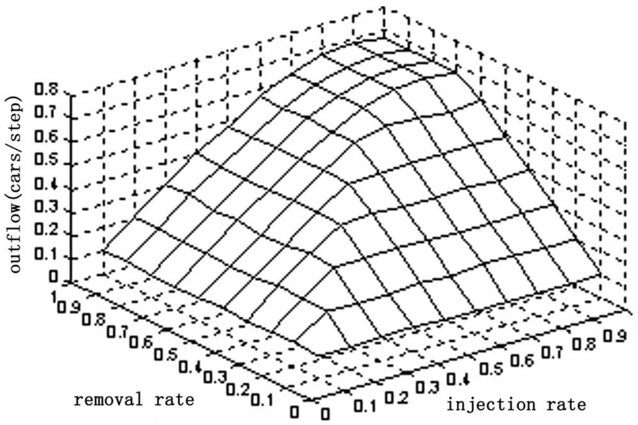

FIG. 12. Outflow in dependence on the injection rate and the removal rate ($v_{max}=5$) (standard injection rule and the new removal rule).

(3) "Mixed"—$t-t_i>1$ and c has experienced IPSD.

Then, this theorem will be proved if we show that s-IPSD($s<4$) can be produced by none of these three ways.

For the direct hit way, then $t-t_i=1$ and $v_{t+1}[n^{-}(c)]=g^{-}(c)=v_t(c)-1$; thus we have $v_t(c)>v_{t+1}[n^{-}(c)]$. That implies $g^{+}_{t+1}[n^{-}(c)]\geq v_t(c)>v_{t+1}[n^{-}(c)]$. Therefore, $n^{-}(c)$ is free driving at time $t+1$, which is contrary to the assumption. The conclusion is that IPSD will be produced by direct hit.

For the acceleration way, proof of inexistence of 3-IPSD is given first. If 3-IPSD happens, then car c must satisfy $v_t(c)<3$. The situation can be divided into two cases. In the system with $v_{max}>2$, it is sure that $t-t_i=2$ [because if $t-t_i>2$, then $v_t(c)\geq 3$]. To ensure the conditions $v_t(c)<3$ and $t-t_i=2$, it is obvious to see that $v_t(c)=g^{-}_t(c)=2$. Thus, $n^{-}(c)$ will not have 3-IPSD at time $t+1$, which is contrary to the assumption. In the case of the system with $v_{max}\leq 2$, because $v_{max}<3$, definitely 3-IPSD will not happen.

We can also prove that 1,2-IPSD cannot be produced by acceleration using same approach shown above. Thus s-IPSD($s<4$) is not producible for the acceleration way.

For the mixed way, we also prove that 3-IPSD does not exist first. Suppose that until time t, c has been slowed down for several times, and the minimum degree among these IPSDs is d_{min}. Notice that $v_t(c)<3$; thus, $d_{min}\leq v_t(c)<3\Rightarrow d_{min}\leq 2$. That means that if 3-IPSD is produced by the mixed way, it requires that s-IPSD($s<3$) happen first.

Recall that we have proved that s-IPSD($s<3$) could be produced by neither of other two ways. Thus, only mixed way can make s-IPSD($s<3$) happen. Using the same approach, a sequence of recursive conclusions is achieved, which causes contradiction: to generate 3-IPSD, there must exist 2-IPSD. Generating 2-IPSD requires 1-IPSD, and so on. Finally, we find that the reason for all s-IPSD($s<4$) is 0-IPSD. But 0-IPSD does not exist in DNS; it follows that 3-IPSD cannot be generated by mixed way. In the same way, we can prove that 1,2-IPSD cannot be produced by mixed way.

It is now obvious that the theorem holds. The minimum degree of IPSD in DNS is 4; as a consequence, if $v_{max}\leq 4$, there are no IPSDs at all.

APPENDIX B: PROOF OF THEOREM 2

Lemma B1. c is a car on the road. If c has IPSD at time t, then $l_{t+1}(c)\geq 8$.

Proof. We have already known that if car c has IPSD at time t, then $v_{t+1}(c)\geq 4$. Moreover the car at site 4 has possibly IPSD. These imply that if car c has IPSD at time t, then $l_{t+1}(c)>8$.

Consider the situation illustrated in Fig. 2. $l_t(c)$ and $v_{t+1}(c)$ could be 4 simultaneously; thus we complete the proof, $l_{t+1}(c)\geq 8$.

Theorem 2. In DNS, suppose that the state of system at time t is S_t and no car is injected, there is only one possible value for S_{t+1} if $v_{max}\leq 5$.

Proof. Let $L_{t+1}(d)$ denote the set which contains all possible values for the location of a car at time $t+1$, whose location at time t is d. Then the theorem could be proved by showing that the following assertion is true: in DNS($v_{max}\leq 5$), $\forall\, d$ satisfying $0<d<v_{max}+1$, then $|L_{t+1}(d)|=1$ or $v_{max}<\min[L_{t+1}(d)]$.

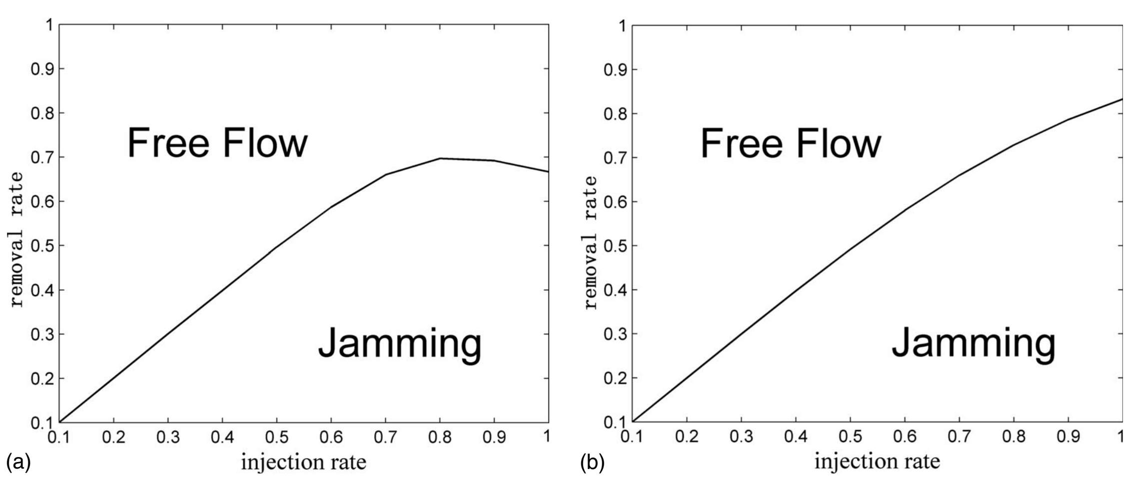

FIG. 13. Phase diagram in dependence on the injection rate and the removal rate ($v_{max}=5$). (a) Standard injection rule. (b) Expanded injection rule.

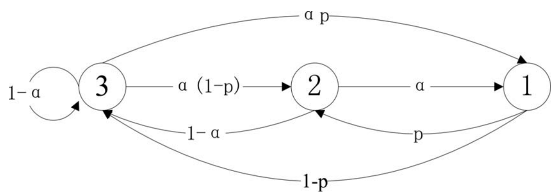

FIG. 14. State transition diagram of the stochastic NS model with $v_{\max}=2$ and standard injection rule.

Another thing to notice is that suppose a car is on site d at time t. Then its velocity is limited within a few values; e.g., if a car is on site 1, then its velocity is definitely 1. This is determined by the rules.

Without consideration of IPSD and $v_{\max}$, we can easily find the possible velocities of cars on the first few sites. Letting $V(d)$ denote the set of possible velocities on site d, we have $V(1)=\{1\}$, $V(2)=\{2\}$, $V(3)=\{2,3\}$, $V(4)=\{4\}$, and $V(5)=\{3,5\}$, and we also have $L_{t+1}(d)=\{d+v+1|v \in V(d)\}$. Then we can discuss it based on the following cases:

(1) $v_{\max}\geq 6$—Since cars on sites 1–3 will never have IPSD, it follows that $V(3)=\{2,3\}$. Thus $L_{t+1}(3)=\{6,7\}$. Notice that $v_{\max}\geq 6$; according to the definition of state, if $S_t=3$ and no car is injected, then $S_{t+1}\in\{6,7\}$.

(2) $v_{\max}=5$—First, without consideration of IPSD, $V(1)=\{1\}$, $V(2)=\{2\}$, $V(3)=\{2,3\}$, $V(4)=\{4\}$, and $V(5)=\{3,5\}$. Therefore, $L_{t+1}(3)=\{6,7\}$ and $L_{t+1}(5)=\{9,10\}$. According to Lemma B1, if c experiences IPSD at time t, then $l_{t+1}(c)\geq 8$. We can see that $v_{\max}<\min(6,7,8,9,10)$.

Therefore, we know that $|L_{t+1}(1)|=|L_{t+1}(2)|=1$ and if $\forall\, d\in\{3,4,5\}$, no matter if affected by IPSD or not, we have $v_{\max}<\min[L_{t+1}(d)]$, which tells us that the assertion holds. For remaining cases $v_{\max}\leq 4$, IPSD is reasonably ignored, and the proof is the same.

(3) $v_{\max}=4$—$V(1)=\{1\}$, $V(2)=\{2\}$, $V(3)=\{2,3\}$, and $V(d\geq 4)=\{4\}$. Thus $L_{t+1}(3)=\{6,7\}$ and $v_{\max}<\min(6,7)$. Moreover if $\forall\, d\in\{1,2,4\}$, then $|L_{t+1}(d)|=1$.

(4) $v_{\max}\leq 3$—$\forall\, d$ satisfying $0<d<v_{\max}+1$, there exists $|V(d)|=1$. Thus, $|L_{t+1}(d)|=1$.

Hence, if the state of system at time t is S_t and no car is injected, there is only one possible value for S_{t+1} if $v_{\max}\leq 5$. The proof is complete.

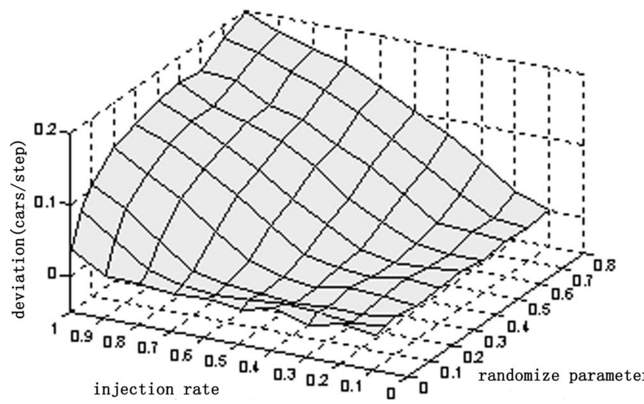

FIG. 15. Deviation between analytical results and simulation results for the stochastic NS model.

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